Math 255A Lecture 28 Notes

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1 Properties of Reflexive Spaces

1.1 Reflexivity of subspaces and the dual space

Last time we proved Kakutani's theorem, that a Banach space B is reflexive if and only if $\{x \in B : \|x\| \leq 1\}$ is compact for $\sigma(B, B^*)$.

Proposition 1.1. *Let B be a reflexive Banach space, and let $M \subseteq B$ be a closed subspace. Then M is reflexive.*

Proof. We have to show that $\{x \in M : \|x\| \leq 1\}$ is compact for $\sigma(M, M^*)$. Now $\sigma(M, M^*)$ agrees with the topology induced on M by $\sigma(B, B^*)$. We can write $\{x \in M : \|x\| \leq 1\} = M \cap \{x \in B : \|x\| \leq 1\}$, where $\{x \in B : \|x\| \leq 1\}$ is compact for $\sigma(B, B^*)$. M is closed and convex, so it is closed for $\sigma(B, B^*)$. Therefore, $M \cap \{x \in B : \|x\| \leq 1\}$ is compact for $\sigma(B, B^*)$, so it is compact for $\sigma(M, M^*)$. By Kakutani's theorem, M is reflexive. $\square$

Corollary 1.1. *A Banach space B is reflexive if and only if B^* is reflexive.*

Proof. ($\implies$): By Banach-Alaoglu, $\{\xi \in B^* : \|\xi\| \leq 1\}$ is compact for $\sigma(B^*, B)$. B is reflexive, so this topology agrees with $\sigma(B^*, B^{**})$, as B is reflexive. This is the weak topology on B^*. By Kakutani's theorem, B^* is reflexive.

($\impliedby$): If B^* is reflexive, by the first part of the proof, B^{**} is reflexive. Now $J : B \to B^{**}$ is isometric, so $J(B) \subseteq B^{**}$ is closed. So $J(B)$ is reflexive. We claim that B is reflexive. In general, if B_1 and B_2 are Banach spaces with B_2 reflexive and there exists $T \in L(B_1, B_2)$ is bijective, then B_1 is reflexive. The adjoint $T^* : B_2^* \to B_1^*$ is bijective; for all $\xi \in B_1^*$, there exists a unique $\eta \in B_2^*$ sicj tjat $\xi = T^*\eta$. Let $y \in B_1^{**}$, and consider (for $\xi \in B_1^*$),

$$\langle \xi, y \rangle = \langle T^*\eta, y \rangle = \langle \eta, T^{**}y \rangle = \langle \eta, x \rangle ,$$

where $x \in B_2$, since B is reflexive (so we can view $T^{**} : B_1^{**} \to B_2$). We get

$$\langle \xi, y \rangle = \langle x, \underbrace{(T^*)^{-1}}_{=(T^{-1})^*} \xi \rangle = \langle \underbrace{T^{-1}x}_{\in B_1}, \xi \rangle.$$

This shows that B_1 is reflexive, and we get that B is reflexive. $\square$

We record the general statement we have proved here for completeness.

Proposition 1.2. *Let B_1 and B_2 be Banach spaces with B_2 reflexive, and let $T \in L(B_1, B_2)$ is bijective. Then B_1 is reflexive.*

Example 1.1. $L^1(\mathbb{R}^n)$ is not reflexive, so $L^\infty(\mathbb{R}^n)$ is not reflexive. This differs from the spaces L^p for $1 < p < \infty$, which are reflexive.

1.2 Compactness properties of the weak topology

Corollary 1.2. *Let B be a reflexive Banach space, and let $K \subseteq B$ be closed, bounded, and convex. Then K is compact for $\sigma(B, B^*)$.*

Proof. K is closed and convex, so K is closed for $\sigma(B, B^*)$. Moreover, $K \subseteq \{x \in B : \|x\| \leq C\}$, which is compact for $\sigma(B, B^*)$. So K is compact. $\square$

Recall: Let B be a separable Banach space, and let $\xi_n \in B^*$ be a such that $\|\xi_n\| \leq C$. Then there exists a subsequence (ξ_{n_k}) which converges in $\sigma(B^*, B)$. We have a similar statement for reflexive Banach spaces which need not be separable.

First, we state a basic fact that we will use.

Proposition 1.3. *Let B be a Banach space. If B^* is separable, then so is B.*

We do not have time to prove this statement, but you can either do the proof yourself or see the proof in Folland's textbook (exercise 25 in chapter 5).

Theorem 1.1. *Let B be a reflexive Banach space, and let (x_n) be a bounded sequence. There exists a subsequence (x_{n_k}) which converges in $\sigma(B, B^*)$.*

Proof. Let $M_0 \subseteq B$ be the space of finite linear combinations of the x_ns. M_0 is separable (using rational coefficients), and so is $M = \overline{M}_0$. Then $x_n \in M$ for all n, and M^* is separable and reflexive. Then $J(M)$ is separable, and $J(M) = M^{**}$. Since M^{**} is separable, we get that M^* is separable. It follows that the weak topology $\sigma(M, M^*)$ on $\{x \in M : \|x\| \leq 1\}$ is metrizable. Thus, $\{x \in M : \|x\| \leq 1\}$ is a compact metric space for $\sigma(M, M^*)$, and there exists a subsequence (x_{n_k}) which converges in $\sigma(M, M^*)$. In other words, $\langle x_{n_k}, \eta \rangle \to \langle x_0, \eta \rangle$ for all $\eta \in M^*$. If $\xi \in B^*$, then $\xi|_M \in M^*$ and so $x_{n_k} \to x_0$ in $\sigma(B, B^*)$. $\square$

Remark 1.1. If B is a Banach space, then B is separable and reflexive if and only if B^* is separable and reflexive.